Fourier analysis of equivariant quantum cohomology I

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Plan

1. Fourier transformation and mirror symmetry
2. shift operator and reduction conjecture
3. Decomposition of quantum cohomology D-modules (spectral decomposition)

$T=(S^1)^\ell \curvearrowright (X,\omega)$: symplectic manifold with Hamiltonian T-action

- D-module version of Teleman's conj (ICM 2014)

$$QH_T^*(X) \overset{FT}{\longleftrightarrow} QH^*(X/\!/T)$$

$\kappa : H_T^*(X) \longrightarrow H^2(X/\!/T)$

$\downarrow$

equiv para λ_i ...... quantum connection $z\nabla_{q_i\partial_{q_i}}$ (roughly $\kappa(\lambda_i)$-direction)

shift operator $\mathbb{S}_i$ ..... Novikov variable q_i

(Seidel map)

$i=1,2,\cdots,\ell$

z : S^1-equiv para rotating $\mathbb{P}^1$

$$[\lambda_i, \mathbb{S}_j] = z\delta_{ij}\mathbb{S}_j \qquad [q_i, zq_j\frac{\partial}{\partial q_j}] = z\delta_{ij}q_j$$

$$\mathcal{S}\circ\lambda = (\lambda - z)\circ\mathcal{S}$$

§ Fourier transformation of symplectic volume

$T \curvearrowright (X,\omega)$ Hamiltonian action

- $H_T^*(X) := H^*(X \times_T ET)$ $\quad ET \to BT$ universal T-bdle

$$\cong H^*(X)\otimes \underbrace{H^*(BT)}_{H_T^*(pt)} \quad (\text{non-canonically})$$

- Cartan model

$$\Big(\Omega^*(X)^{\mathcal{S}'}[\lambda],\ d - \underbrace{\lambda\cdot z}_{\sum \lambda_j z_j}\Big)$$

computes $H_T^*(X)$

- $\mu : M \longrightarrow \mathrm{Lie}(T)^* \cong \mathbb{R}^\ell$

moment map of T-action

- $\hat{\omega} := \omega - \lambda\cdot\mu$

Duistermaat-Heckman form

(equivariantly closed 2-form)

$[\hat{\omega}] \in H_T^2(X)$

$$\int_X e^{\hat{\omega}} = \int_X e^{-\lambda\cdot\mu}\frac{\omega^n}{n!} = \int_{t\in \mathrm{Lie}(T)^*} e^{-\lambda\cdot t}\left(\int_{\vec{\mu}^{-1}(t)/T} e^{\omega_{\mathrm{red}}}\right)dt$$

$\boxed{\text{equivariant volume}}$ $\xleftrightarrow{\ \mathrm{FT}\ }$ $\boxed{\text{volume of } X/\!\!/_t T = \vec{\mu}^{-1}(t)/T}$

function of λ

ω_{red} : reduced symplectic form

<u>Example</u> $X = \mathbb{C}^n \curvearrowleft T = S^1$ diagonal action

function of t : stability

- $\displaystyle\int_X e^{\hat{\omega}} = \int_{\mathbb{C}^n} e^{-\lambda\mu}\, d\mathrm{vol} = \frac{1}{\lambda^n}$

  ☺ $\mu(z) = \frac{1}{2}\sum_{j=1}^n |z_j|^2$

- $\displaystyle\int_{\mathbb{P}^{n-1} = \mu^{-1}(t)/S^1} e^{\omega_{red}} = \int_{\mathbb{P}^{n-1}} \frac{\omega_{red}^{n-1}}{(n-1)!} = \frac{t^{n-1}}{(n-1)!}$ for $t>0$

  ☺ $[\omega_{red}] = t\cdot h$

<u>Rem</u> (inverse FT : Jeffrey - Kirwan residue)

$$\mathrm{vol}(X/\!\!/_t T) = \int_{i\mathbb{R}^l} e^{\lambda t}\, \mathrm{vol}_T(X)\, d\lambda = \sum_F \mathrm{JKRes}_{\lambda=0}\left(e^{\lambda t}\int_F \frac{e^{\hat{\omega}}}{e_T(N_F)}\, d\lambda\right)$$

sum over fixed comp

$$\sum_{F \subset X^T} \int_F \frac{e^{\hat{\omega}}}{e_T(N_F)}$$

Sum of residues

§ Fourier transformation of "quantum" volume. (Givental's heuristics)

- $\widetilde{\mathcal{L}X}$ = universal covering of free loop space
- $\widetilde{\mathcal{L}X}_+ \subset \widetilde{\mathcal{L}X}$ $\frac{\infty}{2}$ cycle

• $\widetilde{\mathcal{L}X}$ = universal covering ...

(assume $\pi_1(X) = \{1\}$)

$\Vert$

$\left\{ \begin{array}{c} \text{holomorphic discs} \\ D^2 \to X \end{array} \right\}$

corresponding to 1
$\wedge$
$QH(X)$

$\left(\underline{\text{Recall}} \quad QH^*(X) = H^{\frac{\infty}{2}+*}(\widetilde{\mathcal{L}X}) \right.$
$\left. \qquad \text{symplectic Floer theory} \right)$

• Givental's path integral

$$\boxed{\int_{\widetilde{\mathcal{L}X}_+} e^{(\Omega - zA)/z}}$$

Ω : symplectic form on $\widetilde{\mathcal{L}X}$ $\left(= \frac{1}{2\pi}\int_0^{2\pi} ev_\theta^* \omega \, d\theta \right)$

$A : \widetilde{\mathcal{L}X} \to \mathbb{R}$ action functional

z : S^1- equivariant parameter
$\quad \Vert$
$\quad S^1_{\text{loop}}$

$\Vert$ heuristic calc
by localization

"quantum volume"

$$\Pi_X := \int_X J_X(-z) \cup z^{n-\frac{\deg}{2}} z^{c_1(X)} \hat{\Gamma}_X$$

$\leftarrow$ comes from $\dfrac{1}{e_{S^1}(N^+)}$

N^+ : positive normal bdl of $X \subset \widetilde{\mathcal{L}X}$

where $$J_X(z) = e^{\omega/z}\left(1 + \sum_{\substack{d\neq 0 \\ i}} \left\langle \frac{\phi^i}{z(z-\psi)} \right\rangle_{0,1,d} e^{\omega\cdot d} \phi_i \right)$$

small J- function

$$\sim e^{\omega/z}$$

<u>Conj (naive)</u> $\quad \Pi_X^{\text{equiv}}$ and $\Pi_{X/\!/T}$ are related by FT

fcn of λ fcn of t

↖ we may need a bulk deformation

Ex 1 $X = \mathbb{C}^n \curvearrowleft S^1$ diagonal

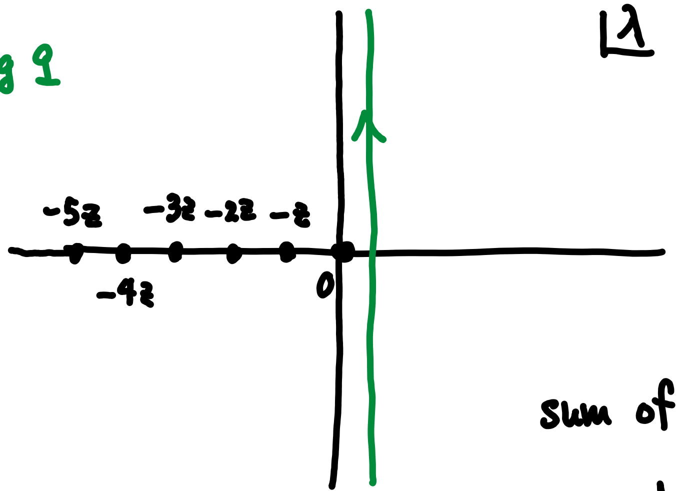
- $$\Pi_X^{eq} = \int_{\mathbb{C}^n} z^{n-\frac{\deg}{2}} z^{n\lambda} \Gamma(1+\lambda)^n = z^{n\lambda/z} \Gamma\left(\frac{\lambda}{z}\right)^n \sim \frac{1}{\lambda^n}$$

- $$\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} \Pi_X^{eq} \, e^{-\lambda \log q / z} \, d\lambda \qquad t = -\log q$$

$$\| $$

$$\int z^{n\lambda/z} \Gamma\left(\frac{\lambda}{z}\right)^n q^{-\lambda/z} \, d\lambda$$

Mellin-Barnes integral



sum of residues at $-dz$

$d = 0, 1, 2, \ldots$

$$\|$$

$$2\pi i z \int_{\mathbb{P}^{n-1}} J_{\mathbb{P}^{n-1}}(-z) \cup \left(z^{n-1-\frac{\deg}{2}} z^{c_1(\mathbb{P}^{n-1})} \hat{\Gamma}_{\mathbb{P}^{n-1}} \right) = 2\pi i z \, \Pi_{\mathbb{P}^{n-1}} \quad \checkmark$$

Ex 2. X : (weak) Fano toric manifold / orbifold $\curvearrowleft T = (S^1)^n$ $\quad n = \dim X$

Thm (I '09)

$$\boxed{\Pi_X^{eq} = \int_{(\mathbb{R}_{>0})^n} \underbrace{e^{-W(x)/z}}_{\Pi_{pt}(W(x))} \cdot e^{\lambda \log x / z} \, \frac{dx}{x}}$$

$W(x)$: mirror LG model for X (Givental - Hori-Vafa)

Note $X /\!/ T = pt$ has the bulk-deformed J-function $J_{pt}(t, -z) = e^{-t/z}$ with $t \in H^0(pt) = \mathbb{C}$

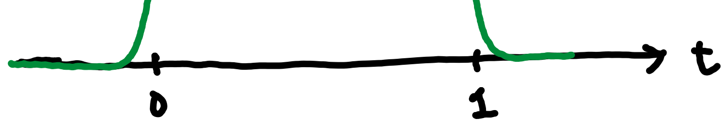
Note mirror variable x_i : (dual to λ) corresponds to the Seidel (shift) operator (Teleman)

Ex $X = \mathbb{P}^1 \curvearrowleft S^1$

$$W(x) = x + \frac{Q}{x}, \quad e^{-W(x)/z} = e^{-\left(x + \frac{Q}{x}\right)/z} = e^{-(Q^t + Q^{1-t})/z}$$

with $x = Q^t$

For $0 < Q \ll 1$, $z > 0$



graph of $e^{-W(x)/z}$

$Q \to 0$ limit

Duistermaat-Heckman measure

$\mu_*(\omega) \qquad \mu : \mathbb{P}^1 \longrightarrow \mathbb{R}$

§ Shift operator for vector spaces

Seidel representation ('97)

$k \in \mathrm{Hom}(S^1, T)$ gives a map $\quad \mathcal{L}X \to \mathcal{L}X$

$(k \in \Omega\,\mathrm{Ham}(X))$ $\qquad \gamma(e^{i\theta}) \mapsto k(e^{i\theta})\,\gamma(e^{i\theta})$

more generally

$\rightsquigarrow \begin{cases} S^k : QH_T^*(X) \circlearrowleft \\ \quad \text{Seidel rep} \\ \mathbb{S}^k : QDM_T(X) \circlearrowleft \\ \quad \text{shift operator} \end{cases}$

(Okounkov-Pandharipande)

<u>Example</u> $X = \mathbb{C}^n \curvearrowleft T = S^1$ diagonal

$$\widetilde{\mathcal{L}X} = \mathcal{L}X = \left\{ \sum_{m \in \mathbb{Z}} a_m e^{im\theta} \;\middle|\; a_m \in \mathbb{C}^n \right\}$$
$$\cup$$
$$\mathcal{L}X_+ = \left\{ \sum_{m=0}^{\infty} a_m e^{im\theta} \right\}$$

($\infty/2$ - fundamental cycle)

<u>Want</u>

$$\mathbb{S} : H_{T \times S^1}^{\frac{\infty}{2}+*}(\widetilde{\mathcal{L}X}) \circlearrowleft$$
$$\cup$$
$$[\mathcal{L}X_+]$$

$$\mathbb{S} : \gamma(e^{i\theta}) \mapsto e^{i\theta} \cdot \gamma(e^{i\theta})$$

Rem This map is not $(T \times S^1)$-equiv

But is equiv w.r.t. the grp action

$T \times S^1 \to T \times S^1 \quad (\lambda, z) \mapsto (\lambda z, z)$

$$\mathbb{S} \circ \lambda = (\lambda - z) \circ \mathbb{S}$$

<u>$\frac{\infty}{2}$ class</u>

$$[\mathcal{L}X_+] = \{a_{-1} = a_{-2} = \cdots = 0\} \text{ in } \mathcal{L}X$$

$$= \prod_{m=1}^{\infty} (\lambda + mz)^n$$

(a_{-m} has $(T \times S^1)$ wt $\lambda + mz$)

$$\therefore \quad \mathbb{S}\,[\mathcal{L}X_+] = \lambda^n \cdot [\mathcal{L}X_+]$$

$\rightsquigarrow$ relation $\quad \mathbb{S} \cdot 1 = \lambda^n \cdot 1 \quad$ in $QDM_T(X)$ equiv quantum D-mod

$\left(\cong H^{\frac{\infty}{2}}_{T \times S^1}(\widetilde{\mathcal{L}X}) \right)$

<u>Fourier duality</u> : $QDM_T(\mathbb{C}^n) \cong QDM(\mathbb{P}^{n-1})$

$$(\mathbb{S} - \lambda^n) \cdot 1 = 0 \longleftrightarrow \left(Q - (z \nabla_{Q\partial/\partial Q})^n \right) 1 = 0$$

quantum differential eqn of $\mathbb{P}^{n-1}$

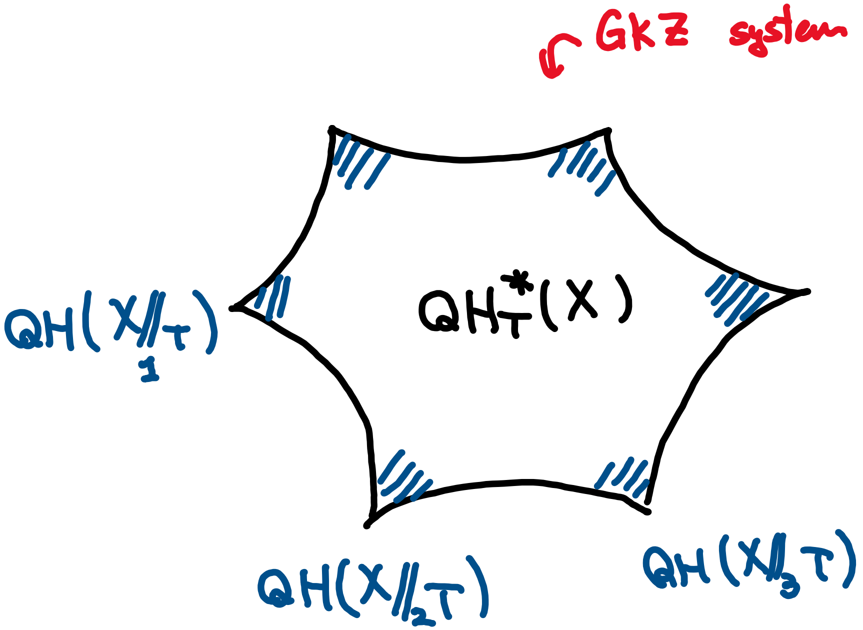
<u>Rem</u> More generally, $\mathbb{C}^n \curvearrowleft T = (S^1)^\ell$ linear representation

with weights $D_1, \cdots, D_n \in \bigoplus_{j=1}^{\ell} \mathbb{Z}\lambda_j$

$\rightsquigarrow \quad \mathbb{S}^k \circlearrowright \underset{\underset{\displaystyle QDM_T(\mathbb{C}^n)}{\shortparallel}}{H^*_{T\times S^1}(\mathbb{C}^n)}$ is given by $\quad \mathbb{S}^k = \left(\prod_{j=1}^{n} \dfrac{\prod_{c=-\infty} D_j + cz}{\prod_{c=-\infty}^{-D_j\cdot k} D_j + cz} \right) e^{-kz\partial_\lambda}$

<u>GKZ relation</u>

$$\left[\prod_{i:\, D_i\cdot k>0} \ \prod_{c=0}^{D_i\cdot k-1} (D_i - cz) - \mathbb{S}^k \prod_{i:\, D_i\cdot k<0} \ \prod_{c=0}^{-D_i\cdot k-1} (D_i - cz) \right] \cdot 1 = 0$$



"Global Kähler moduli space"

for $X/\!\!/T$